4: Aspects of Ambisonic Systems

This thesis does not include any detailed discussion of ambisonic reproduction systems or of the psychoacoustic theories on which ambisonic decoders are based. A significant feature of ambisonic technology is that recording and reproduction are decoupled to the maximum extent possible [65]; hence, soundfield microphones can be treated without reference to reproduction systems. This decoupling is achieved by defining the way in which directional information is encoded in ambisonic systems, using the B-format signal set. So long as a soundfield microphone provides output signals conforming to the B-format specification, there is no need to consider how those signals will be used by a reproduction system (and conversely, such a system can be designed to operate from B-format signals without consideration of the origin of those signals).

4.1: The B-Format Signal Set

The B-format signals are usually defined in terms of spherical harmonics [31] [40]; it follows from the analysis presented in Chapter 3 that they can also be expressed in terms of derivatives of sound pressure. The nth-order B-format signal set consists of the signals which would be obtained from $(n+1)^2$ coincident microphones having polar patterns corresponding to the $(n+1)^2$ linearly independent spherical harmonics of all orders up to and including n. A signal set of any order can be extended to a higher order merely by augmenting it with additional signals; it is not necessary to change any of the existing signals [5] [40]. An nth-order soundfield microphone is distinguished by its ability to provide outputs which are the nth-order B-format signals.

As discussed in Chapter 3, a set of nine signals corresponding to the outputs of microphones with polar patterns given by the spherical harmonics of order up to and including two, which are the second-order B-format signals, may be combined to synthesise the output of an arbitrary microphone of order two or lower in any orientation. This result may be generalised: the nth-order B-format signals may be combined such as to produce the output of an arbitrary microphone of order n or lower in any orientation. This provides one justification for the claim that the second-order soundfield microphone captures all the information up to second order present in the original sound field.

In the following section it is demonstrated that the second-order B-format signals contain sufficient information to reconstruct a plane wave up to the second order terms in certain series expansions; this provides a second way in which second-order B-format may be said to contain complete information up to second order.

The signal designations used in this thesis, shown in table 4.1, are those proposed by Richard Furse and Dave Malham [36]. First-order B-format consists of the zeroth-order signal *W* and the three first-order spherical harmonic component signals *X*, *Y* and *Z*. The five second-order spherical harmonic component signals are added to form second-order B-format.

Signal Designation	Polar Pattern	Derivative Expression (Neglecting Equalisation)
W	$1/\sqrt{2}$	p
X	$\cos(q)\cos(f)$	$\frac{\partial p}{\partial x}$
Y	$\sin(q)\cos(f)$	$\frac{\partial p}{\partial y}$
Z	$\sin(f)$	$\frac{\partial p}{\partial z}$
R	$\frac{1}{2}\left(3\sin^2(f)-1\right)$	$\frac{1}{2}\left(3\frac{\partial^2 p}{\partial z^2}+k^2 p\right)$
S	$\cos(q)\sin(2f)$	$2\frac{\partial^2 p}{\partial x \partial z}$
T	$\sin(q)\sin(2f)$	$2\frac{\partial^2 p}{\partial y \partial z}$
U	$\cos(2q)\cos^2(f)$	$\frac{\partial^2 p}{\partial x^2}-\frac{\partial^2 p}{\partial y^2}$
V	$\sin(2q)\cos^2(f)$	$2\frac{\partial^2 p}{\partial x \partial y}$

Table 4.1: Furse-Malham Second-Order B-Format Signal Set

The 3 dB attenuation of *W* compared to the other signals was originally included in the definition of first-order B-format to approximately equalise the energy levels in *W* and the three first-order component signals, and is maintained in the second-order B-format definition for reasons of backward compatibility. Comparison of the remaining signals with the spherical harmonics given in table 2.3 shows that the associated polar responses have been normalised to have the same maximum value, eliminating the scaling factors which appear in the definitions of the spherical harmonics themselves.

4.2: Analysis of Second-Order Reproduction

Although ambisonic recording can be understood without reference to the way in which the B-format signals are used by a playback system, appreciation of the information carried by these signals is enhanced by some knowledge of the reproduction process.

The analysis presented in this section is entirely at the physical acoustic level; no psychoacoustic considerations are included. It may reasonably be assumed that, if the original sound field could be physically reconstructed, then all of the spatial information originally present would be conveyed to the listener. However, such exact physical reconstruction can not be accomplished by a system employing a practical number of channels and loudspeakers. Ambisonic decoding is based on approximate reconstruction of the sound field only at low frequencies, if at all; at higher frequencies, decoding schemes are intended to conform to psychoacoustic requirements which are not in fact satisfied by attempts to physically recreate the sound field [31] [32] [41] [65]. Various decoding algorithms have been proposed [27] [47] [58] [65]. The purpose of this section is to demonstrate that certain information is carried by the second-order B-format signals, not to consider how this information can most effectively be used to create the desired spatial effect for the listener.

4.2.1: Spherical Harmonic Matching Conditions

In [5], [6] and [90], the ability of an ambisonic system to reproduce a plane wave is analysed in terms of the so-called “spherical harmonic matching conditions”. The method of analysis utilised in these references is applicable to pantophonic ambisonic systems of arbitrary order, although the authors concentrate primarily on first-order and second-order systems. Here, a parallel analysis is presented which is specifically concerned with second-order periphonic systems; the approach can again be generalised to systems of higher order, but this is unsurprisingly more complicated than in the pantophonic case. (It should be noted that the notation employed here does not match that used in the cited references).

Consider a plane wave propagating from a direction (q, f) with wave incidence vector

$$\tilde{\mathbf{k}} = k\begin{bmatrix} \cos(q)\cos(f) \\ \sin(q)\cos(f) \\ \sin(f) \end{bmatrix} \qquad (4.1)$$

Let A and y be respectively the amplitude of the wave and its phase at the origin of the coordinate frame. Consider a point located a distance *r* from the origin in a direction (q', f'); let the position vector

$$\mathbf{r} = r\begin{bmatrix} \cos(q')\cos(f') \\ \sin(q')\cos(f') \\ \sin(f') \end{bmatrix} \qquad (4.2)$$

(see figure 4.1).

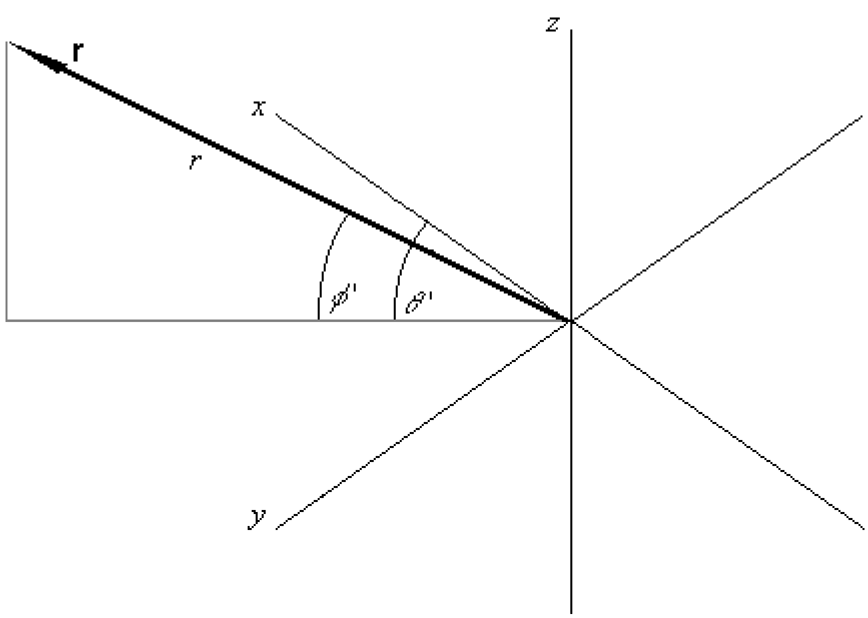

Figure 4.1: Position Vector **r**

The sound pressure at this point due to the incident wave

$$
\begin{aligned}
p_i &= Ae^{j(wt+y+\tilde{\mathbf{k}}\cdot\mathbf{r})} \\
&= Ae^{j(wt+y+kr\cos(q)\cos(f)\cos(q')\cos(f')+kr\sin(q)\cos(f)\sin(q')\cos(f')+kr\sin(f)\sin(f'))} \\
&= Ae^{j(wt+y)}e^{jkr[\cos(q)\cos(f)\cos(q')\cos(f')+\sin(q)\cos(f)\sin(q')\cos(f')+\sin(f)\sin(f')]} \\
&= Ae^{j(wt+y)}e^{jkr[(\cos(q)\cos(q')+\sin(q)\sin(q'))\cos(f)\cos(f')+\sin(f)\sin(f')]} \\
&= Ae^{j(wt+y)}e^{jkr[\cos(q-q')\cos(f)\cos(f')+\sin(f)\sin(f')]}
\end{aligned}
\tag{4.3}
$$

Now

$$
\begin{aligned}
\cos(q-q')\cos(f)\cos(f')+\sin(f)\sin(f') &= \frac{1}{kr}\tilde{\mathbf{k}}\cdot\mathbf{r} \\
&= \left(\frac{1}{k}\tilde{\mathbf{k}}\right)\cdot\left(\frac{1}{r}\mathbf{r}\right)
\end{aligned}
\tag{4.4}
$$

i.e., the scalar product of the unit vectors $(1/k)\tilde{\mathbf{k}}$ and $(1/r)\mathbf{r}$. It is therefore equal to the cosine of the angle between these vectors; let this angle be z (see figure 6.2), then

$$\cos(z)=\cos(q-q')\cos(f)\cos(f')+\sin(f)\sin(f') \tag{4.5}$$

and

$$p_i = Ae^{j(wt+y)}e^{jkr\cos(z)} \tag{4.6}$$

From equation (2.15), we have

$$e^{jkr\cos(z)}=\sum_{n=0}^{\infty} j^n(2n+1)P_n\big(\cos(z)\big)j_n(kr) \tag{4.7}$$

and so we may write

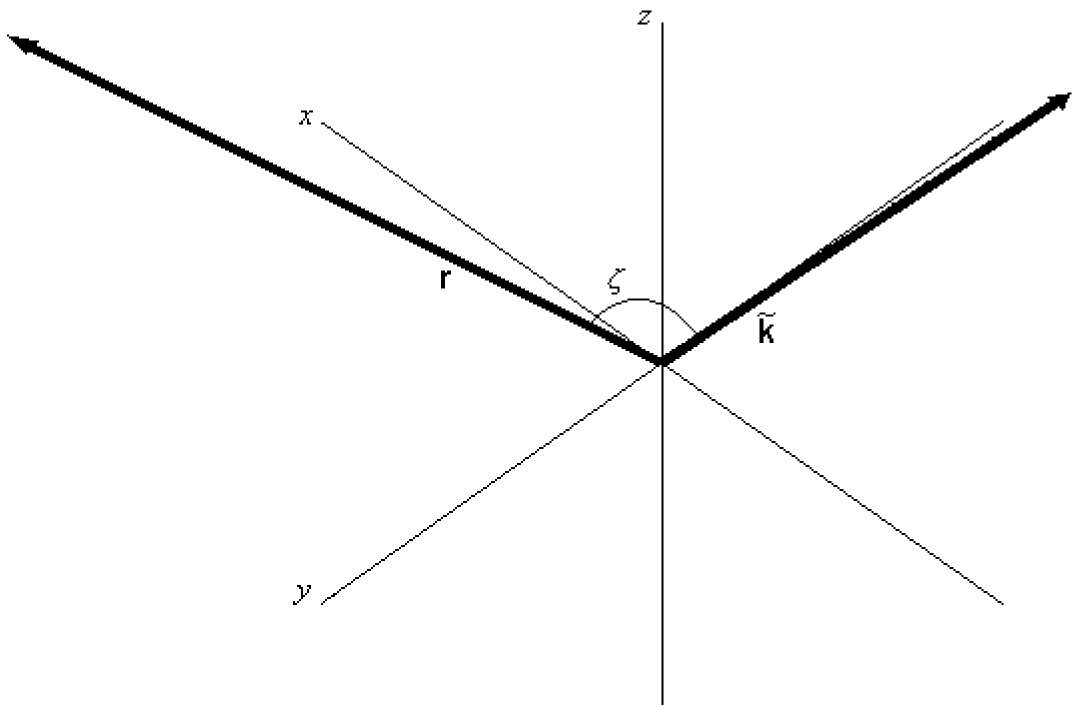

Figure 4.2: Angle Between Vectors **r** and $\tilde{\mathbf{k}}$

$$\begin{aligned} p_i &= Ae^{j(wt+y)}\sum_{n=0}^{\infty} j^n(2n+1)P_n\big(\cos(z)\big)j_n(kr) \\ &= Ae^{j(wt+y)}\sum_{n=0}^{\infty} j^n(2n+1)P_n\big(\cos(q-q')\cos(f)\cos(f')+\sin(f)\sin(f')\big)j_n(kr) \end{aligned} \qquad (4.8)$$

Consider now an array of N loudspeakers, equidistant from the origin. The driving signal for the mth loudspeaker is designated L_m. The output of each loudspeaker is assumed to be a plane wave such that, if

$$L_m = A_m e^{j(wt+y)} \qquad (4.9)$$

then the resulting pressure at the centre of the loudspeaker array (the origin) is

$$p_m = A_m e^{j(wt+y)} \qquad (4.10)$$

This implies that the frequency response function relating the electrical input and acoustic output of each loudspeaker is

$$L(w) = e^{jwD/c} \tag{4.11}$$

where D is the distance of each loudspeaker from the origin, which in turn implies that the loudspeakers are non-causal. This is purely a convention that simplifies the analysis by enabling us to ignore the time delay resulting from the finite time required for sound to propagate from a loudspeaker to the array centre. In reality, of course, this delay is present, as in fact is the arbitrary time delay which may exist between the recording and playback of an acoustic event. The actual comparison which is important is between the original sound field and that generated by the reproduction system when each is expressed relative to its own temporal origin. However, for the purposes of the present analysis it is convenient to describe both sound fields in terms of the same spatial and temporal coordinates, and it is simpler to assume non-causal loudspeakers than to include a linear phase lag factor which will in any case only be neglected at a later stage.

Let the *m*th speaker be positioned in a direction (q_m, f_m), then the sound pressure at the point **r** due to the plane wave output of the *m*th speaker can be written

$$p_m = A_m e^{j(wt+y)} \sum_{n=0}^{\infty} j^n (2n+1) P_n\left(\cos(z_m)\right) j_n(kr) \tag{4.12}$$

where

$$\cos(z_m) = \cos(q_m - q')\cos(f_m)\cos(f') + \sin(f_m)\sin(f') \tag{4.13}$$

and the total sound pressure due to the outputs of all loudspeakers in the array is

$$\begin{aligned} p &= \sum_{m=1}^{N}\left[A_m e^{j(wt+y)} \sum_{n=0}^{\infty} j^n (2n+1) P_n\left(\cos(z_m)\right) j_n(kr) \right] \\ &= \sum_{n=0}^{\infty}\left[j^n (2n+1)\, j_n(kr) e^{j(wt+y)} \sum_{m=1}^{N} A_m P_n\left(\cos(z_m)\right) \right] \end{aligned} \tag{4.14}$$

By comparing equations (4.8) and (4.14), we see that $p = p_i$ if, for all n,

$$Ae^{j(wt+y)} j^n (2n+1) P_n(\cos(z)) j_n(kr) = j^n (2n+1) j_n(kr) e^{j(wt+y)} \sum_{m=1}^{N} A_m P_n(\cos(z_m)) \quad (4.15)$$

or, cancelling the common factors $j^n(2n+1)$, $j_n(kr)$ and $e^{j(wt+y)}$,

$$AP_n(\cos(z)) = \sum_{m=1}^{N} A_m P_n(\cos(z_m)) \quad (4.16)$$

Now, for each *n*, $P_n(\cos(z))$ and $P_n(\cos(z_m))$ can be expressed as linear combinations of standard *n*th-order spherical harmonics.

For $n = 0$, we have

$$AP_0(\cos(z)) = \sum_{m=1}^{N} A_m P_0(\cos(z_m)) \quad (4.17)$$

which simplifies, since $P_0(m) = 1$, to

$$A = \sum_{m=1}^{N} A_m \quad (4.18)$$

For $n = 1$, equation (4.16) becomes

$$AP_1(\cos(z)) = \sum_{m=1}^{N} A_m P_1(\cos(z_m)) \quad (4.19)$$

or, since $P_1(m) = m$,

$$A\cos(z) = \sum_{m=1}^{N} A_m \cos(z_m) \quad (4.20)$$

which may be expanded to give

$$
\begin{aligned}
&A\left[\cos(q - q')\cos(f)\cos(f') + \sin(f)\sin(f')\right] \\
&\qquad = \sum_{m=1}^{N} A_m \left[\cos(q_m - q')\cos(f_m)\cos(f') + \sin(f_m)\sin(f')\right] \\
&A\left[\cos(q')\cos(f')\cos(q)\cos(f) + \sin(q')\cos(f')\sin(q)\cos(f) + \sin(f)\sin(f')\right] \\
&\qquad = \sum_{m=1}^{N} A_m \left[\cos(q')\cos(f')\cos(q_m)\cos(f_m) + \sin(q')\cos(f')\sin(q_m)\cos(f_m) \right. \\
&\qquad\qquad \left. + \sin(f')\sin(f_m)\right]
\end{aligned}
\qquad (4.21)
$$

This is satisfied if

$$
\begin{aligned}
A\cos(q')\cos(f')\cos(q)\cos(f) &= \sum_{m=1}^{N} A_m \cos(q')\cos(f')\cos(q_m)\cos(f_m) \\
&= \cos(q')\cos(f')\sum_{m=1}^{N} A_m \cos(q_m)\cos(f_m)
\end{aligned}
\qquad (4.22a)
$$

$$
\begin{aligned}
A\sin(q')\cos(f')\sin(q)\cos(f) &= \sum_{m=1}^{N} A_m \sin(q')\cos(f')\sin(q_m)\cos(f_m) \\
&= \sin(q')\cos(f')\sum_{m=1}^{N} A_m \sin(q_m)\cos(f_m)
\end{aligned}
\qquad (4.22b)
$$

$$
\begin{aligned}
A\sin(f')\sin(f) &= \sum_{m=1}^{N} A_m \sin(f')\sin(f_m) \\
&= \sin(f')\sum_{m=1}^{N} A_m \sin(f_m)
\end{aligned}
\qquad (4.22c)
$$

i.e., if

$$A\cos(q)\cos(f) = \sum_{m=1}^{N} A_m \cos(q_m)\cos(f_m) \qquad (4.23a)$$

$$A\sin(q)\cos(f) = \sum_{m=1}^{N} A_m \sin(q_m)\cos(f_m) \qquad (4.23b)$$

$$A\sin(f) = \sum_{m=1}^{N} A_m \sin(f_m) \qquad (4.23c)$$

Finally, for $n = 2$ we obtain

$$AP_2\left(\cos(z)\right) = \sum_{m=1}^{N} A_m P_2\left(\cos(z_m)\right) \qquad (4.24)$$

which, by appropriate substitution for $P_2(m)$, becomes

$$\begin{aligned}
&A\frac{1}{2}\left(3\cos^2(z)-1\right)=\sum_{m=1}^{N}A_m\,\frac{1}{2}\left(3\cos^2(z_m)-1\right)\\
&A\Big[\left(1-\tfrac{3}{2}\cos^2(f')\right)\tfrac{1}{2}\left(3\sin^2(f)-1\right)+\sin(f')\cos(f')\cos(q')\tfrac{3}{2}\cos(q)\sin(2f)\\
&\qquad+\left(\tfrac{1}{2}\cos^2(f')\cos^2(q')-\tfrac{1}{4}\cos^2(f')\right)3\cos(2q)\cos^2(f)\\
&\qquad+\sin(f')\cos(f')\sin(q')\tfrac{3}{2}\sin(q)\sin(2f)\\
&\qquad+\tfrac{1}{2}\cos^2(f')\cos(q')\sin(q')3\sin(2q)\cos^2(f)\Big]\\
&=\sum_{m=1}^{N}A_m\Big[\left(1-\tfrac{3}{2}\cos^2(f')\right)\tfrac{1}{2}\left(3\sin^2(f_m)-1\right)\\
&\qquad+\sin(f')\cos(f')\cos(q')\tfrac{3}{2}\cos(q_m)\sin(2f_m)\\
&\qquad+\left(\tfrac{1}{2}\cos^2(f')\cos^2(q')-\tfrac{1}{4}\cos^2(f')\right)3\cos(2q_m)\cos^2(f_m)\\
&\qquad+\sin(f')\cos(f')\sin(q')\tfrac{3}{2}\sin(q_m)\sin(2f_m)\\
&\qquad+\tfrac{1}{2}\cos^2(f')\cos(q')\sin(q')3\sin(2q_m)\cos^2(f_m)\Big]
\end{aligned}\tag{4.25}$$

which is satisfied if

$$\begin{aligned}
A\left(1-\tfrac{3}{2}\cos^2(f')\right)\tfrac{1}{2}\left(3\sin^2(f)-1\right)&=\sum_{m=1}^{N}A_m\left(1-\tfrac{3}{2}\cos^2(f')\right)\tfrac{1}{2}\left(3\sin^2(f_m)-1\right)\\
&=\left(1-\tfrac{3}{2}\cos^2(f')\right)\sum_{m=1}^{N}A_m\,\tfrac{1}{2}\left(3\sin^2(f_m)-1\right)
\end{aligned}\tag{4.26a}$$

$$\begin{aligned}
&A\sin(f')\cos(f')\cos(q')\tfrac{3}{2}\cos(q)\sin(2f)\\
&\qquad=\sum_{m=1}^{N}A_m\sin(f')\cos(f')\cos(q')\tfrac{3}{2}\cos(q_m)\sin(2f_m)\\
&\qquad=\sin(f')\cos(f')\cos(q')\sum_{m=1}^{N}A_m\,\tfrac{3}{2}\cos(q_m)\sin(2f_m)
\end{aligned}\tag{4.26b}$$

$$\begin{aligned}
&A\left(\tfrac{1}{2}\cos^2(f')\cos^2(q')-\tfrac{1}{4}\cos^2(f)\right)3\cos(2q)\cos^2(f)\\
&\qquad=\sum_{m=1}^{N}A_m\left(\tfrac{1}{2}\cos^2(f')\cos^2(q')-\tfrac{1}{4}\cos^2(f')\right)3\cos(2q_m)\cos^2(f_m)\\
&\qquad=\left(\tfrac{1}{2}\cos^2(f')\cos^2(q')-\tfrac{1}{4}\cos^2(f')\right)\sum_{m=1}^{N}A_m3\cos(2q_m)\cos^2(f_m)
\end{aligned}\tag{4.26c}$$

$$\begin{aligned}
&A\sin(f')\cos(f')\sin(q')\tfrac{3}{2}\sin(q)\sin(2f)\\
&\qquad=\sum_{m=1}^{N}A_m\sin(f')\cos(f')\sin(q')\tfrac{3}{2}\sin(q_m)\sin(2f_m)\\
&\qquad=\sin(f')\cos(f')\sin(q')\sum_{m=1}^{N}A_m\,\tfrac{3}{2}\sin(q_m)\sin(2f_m)
\end{aligned}\tag{4.26d}$$

$$A\tfrac{1}{2}\cos^2(f')\cos(q')\sin(q')3\sin(2q)\cos^2(f)$$
$$=\sum_{m=1}^{N} A_m \tfrac{1}{2}\cos^2(f')\cos(q')\sin(q')3\sin(2q_m)\cos^2(f_m) \qquad (4.26e)$$
$$=\cos^2(f')\cos(q')\sin(q')\sum_{m=1}^{N} A_m \tfrac{1}{2}3\sin(2q_m)\cos^2(f_m)$$

i.e.,

$$A\left(3\sin^2(f)-1\right)=\sum_{m=1}^{N} A_m\left(3\sin^2(f_m)-1\right) \qquad (4.27a)$$
$$A\cos(q)\sin(2f)=\sum_{m=1}^{N} A_m\cos(q_m)\sin(2f_m) \qquad (4.27b)$$
$$A\cos(2q)\cos^2(f)=\sum_{m=1}^{N} A_m\cos(2q_m)\cos^2(f_m) \qquad (4.27c)$$
$$A\sin(q)\sin(2f)=\sum_{m=1}^{N} A_m\sin(q_m)\sin(2f_m) \qquad (4.27d)$$
$$A\sin(2q)\cos^2(f)=\sum_{m=1}^{N} A_m\sin(2q_m)\cos^2(f_m) \qquad (4.27e)$$

Equations (4.18), (4.23) and (4.27) are the spherical harmonic matching conditions up to and including second order; if these are satisfied, then equations (4.8) and (4.14) will be equal up to the terms of second order.

If the loudspeakers are positioned at the vertices of a notional regular dodecahedron, it may be verified by substitution and lengthy algebraic and trigonometric manipulation that these matching conditions are satisfied if the loudspeaker feed signals are

$$L_m=\frac{A}{12}\Big[\sqrt{2}W+3\cos(q_m)\cos(f_m)X+3\sin(q_m)\cos(f_m)Y+3\sin(f_m)Z$$
$$+\frac{15}{4}\cos(2q_m)\cos^2(f_m)U+\frac{15}{4}\sin(2q_m)\cos^2(f_m)V \qquad (4.28)$$
$$+\frac{15}{4}\sin(q_m)\sin(2f_m)T+\frac{15}{4}\cos(q_m)\sin(2f_m)S+\frac{5}{2}\left(3\sin^2(f_m)-1\right)R\Big]$$

where the second-order B-format signals are measured at the origin. Hence, these signals carry all the information required to construct a second-order approximation of a plane wave.

This analysis may be reduced to the pantophonic case by putting $f=0$ and $f_m=0$ for all *m*. Some of the previously obtained matching equations then reduce to $0=0$, since $\sin(0)=0$. The remaining set is

$$A=\sum_{m=1}^{N} A_m \tag{4.29a}$$

$$A\cos(q)=\sum_{m=1}^{N} A_m \cos(q_m) \tag{4.29b}$$

$$A\sin(q)=\sum_{m=1}^{N} A_m \sin(q_m) \tag{4.29c}$$

$$-A=\sum_{m=1}^{N} (-A_m) \tag{4.29d}$$

$$A\cos(2q)=\sum_{m=1}^{N} A_m \cos(2q_m) \tag{4.29e}$$

$$A\sin(2q)=\sum_{m=1}^{N} A_m \sin(2q_m) \tag{4.29f}$$

The fourth equation here is clearly redundant, since it expresses the same condition as the first. The remaining five equations are the spherical harmonic matching conditions for second-order pantophonic systems, as given in [5] and [6].

4.2.2: The Maclaurin Series Interpretation

Let the position vector **r** designate a point close to the origin, and let p_2 be the pressure at **r** expressed as a Maclaurin series truncated to second order:

$$\begin{aligned} p_2 &= p+\mathbf{r}_x\frac{\partial p}{\partial x}+\mathbf{r}_y\frac{\partial p}{\partial y}+\mathbf{r}_z\frac{\partial p}{\partial z} \\ &\quad +\tfrac{1}{2}\mathbf{r}_x{}^2\frac{\partial^2 p}{\partial x^2}+\mathbf{r}_x\mathbf{r}_y\frac{\partial^2 p}{\partial x\partial y}+\mathbf{r}_x\mathbf{r}_z\frac{\partial^2 p}{\partial x\partial z}+\tfrac{1}{2}\mathbf{r}_y{}^2\frac{\partial^2 p}{\partial y^2}+\mathbf{r}_y\mathbf{r}_z\frac{\partial^2 p}{\partial y\partial z}+\tfrac{1}{2}\mathbf{r}_z{}^2\frac{\partial^2 p}{\partial z^2} \\ &= p+\mathbf{r}_x\frac{\partial p}{\partial x}+\mathbf{r}_y\frac{\partial p}{\partial y}+\mathbf{r}_z\frac{\partial p}{\partial z} \\ &\quad +\frac{1}{2}\left(\mathbf{r}_x{}^2\frac{\partial^2 p}{\partial x^2}+2\mathbf{r}_x\mathbf{r}_y\frac{\partial^2 p}{\partial x\partial y}+2\mathbf{r}_x\mathbf{r}_z\frac{\partial^2 p}{\partial x\partial z}+\mathbf{r}_y{}^2\frac{\partial^2 p}{\partial y^2}+2\mathbf{r}_y\mathbf{r}_z\frac{\partial^2 p}{\partial y\partial z}+\mathbf{r}_z{}^2\frac{\partial^2 p}{\partial z^2}\right) \end{aligned} \tag{4.30}$$

(where *p* and all derivatives are, of course, measured at the origin). We may alternatively write

$$
\begin{aligned}
p_2 = {} & p + \mathbf{r}_x \frac{\partial p}{\partial x} + \mathbf{r}_y \frac{\partial p}{\partial y} + \mathbf{r}_z \frac{\partial p}{\partial z} + \mathbf{r}_x \mathbf{r}_y \frac{\partial^2 p}{\partial x \partial y} + \mathbf{r}_x \mathbf{r}_z \frac{\partial^2 p}{\partial x \partial z} + \mathbf{r}_y \mathbf{r}_z \frac{\partial^2 p}{\partial y \partial z} \\
& + \tfrac{1}{2}\mathbf{r}_x{}^2 \frac{1}{2}\left(\nabla^2 p - \frac{2}{3}\left[\frac{1}{2}\left[3\frac{\partial^2 p}{\partial z^2} - \nabla^2 p \right] + \frac{1}{2}\nabla^2 p \right] + \left[\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} \right] \right) \\
& + \tfrac{1}{2}\mathbf{r}_y{}^2 \frac{1}{2}\left(\nabla^2 p - \frac{2}{3}\left[\frac{1}{2}\left[3\frac{\partial^2 p}{\partial z^2} - \nabla^2 p \right] + \frac{1}{2}\nabla^2 p \right] - \left[\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} \right] \right) \\
& + \tfrac{1}{2}\mathbf{r}_z{}^2 \frac{2}{3}\left(\frac{1}{2}\left[3\frac{\partial^2 p}{\partial z^2} - \nabla^2 p \right] + \frac{1}{2}\nabla^2 p \right)
\end{aligned}
\tag{4.31}
$$

If the sound field is assumed to satisfy the Helmholtz equation, this can be rewritten as

$$
\begin{aligned}
p_2 = {} & p + \mathbf{r}_x \frac{\partial p}{\partial x} + \mathbf{r}_y \frac{\partial p}{\partial y} + \mathbf{r}_z \frac{\partial p}{\partial z} + \mathbf{r}_x \mathbf{r}_y \frac{\partial^2 p}{\partial x \partial y} + \mathbf{r}_x \mathbf{r}_z \frac{\partial^2 p}{\partial x \partial z} + \mathbf{r}_y \mathbf{r}_z \frac{\partial^2 p}{\partial y \partial z} \\
& + \tfrac{1}{2}\mathbf{r}_x{}^2 \frac{1}{2}\left(-k^2 p - \frac{2}{3}\left[\frac{1}{2}\left[3\frac{\partial^2 p}{\partial z^2} + k^2 p \right] - \frac{1}{2}k^2 p \right] + \left[\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} \right] \right) \\
& + \tfrac{1}{2}\mathbf{r}_y{}^2 \frac{1}{2}\left(-k^2 p - \frac{2}{3}\left[\frac{1}{2}\left[3\frac{\partial^2 p}{\partial z^2} + k^2 p \right] - \frac{1}{2}k^2 p \right] - \left[\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} \right] \right) \\
& + \tfrac{1}{2}\mathbf{r}_z{}^2 \frac{2}{3}\left(\frac{1}{2}\left[3\frac{\partial^2 p}{\partial z^2} + k^2 p \right] - \frac{1}{2}k^2 p \right)
\end{aligned}
\tag{4.32}
$$

With reference to table 4.1, it may be seen that this equation is expressed entirely in quantities which are represented by the second-order B-format signals; this demonstrates that the second-order B-format signals contain the necessary information to construct an approximation of the original soundfield.

However, this approach also shows clearly that in fact the second-order B-format signal set only carries complete second-order information in cases where the Helmholtz equation is satisfied. Where this is not the case, equations (4.31) and (4.32) are not equivalent, because all the derivatives in equation (4.30) are linearly independent. Under these circumstances, some information about the second-order derivatives of the sound field is not preserved by a B-format recording.